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Controllability and Observability for Flexible Spacecraft

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Current interest in extended sensing and actuation for the control of flexible spacecraft has led to the use of modern multivariable control theory and the associated concepts of controllability and observability. This paper shows how to evaluate these properties on a mode-by-mode basis for flexible spacecraft control analysis. Relatively simple criteria are derived which indicate the degree of controllability (observability) of each mode in simple literal terms. These criteria provide physical insight and practical guidance on the type, number, and positioning of sensors and actuators. The results are interpreted for force and torque actuators and for attitude and deformation measurements. To illustrate these ideas, sample controllability and observability "surfaces" are presented for the Purdue generic flexible spacecraft model.

Introduction

SERIOUS application of "modern" control theory to the design of attitude control systems for real spacecraft has been begun only recently.[1] In handling problems of this nature, it is recognized that vehicle elasticity must be carefully modeled, thus requiring many degrees of freedom in the system model. It is also often necessary (notably in antenna and telescope applications) to control the shape or "figure" of part of the vehicle; tolerances as tight as 1 mm have been mentioned.[2] To meet these control objectives, sufficient sensors and actuators must be placed at appropriate locations. For these reasons, many current spacecraft control problems must be treated as systems with many state variables, many inputs, and many outputs. It is precisely to such problems that "modern" multivariable control theory is addressed.

If sensors and actuators are to be distributed throughout the spacecraft, three fundamental questions are: What types of sensing and actuation are required? What is the least number required? and Where should they be placed? To answer the question of sensor and actuator positioning, an analytical criterion is needed to facilitate a quantitative comparison of locations. One such criterion, the modal controllability, is introduced in this paper for actuator placement, together with an analogous criterion, the modal observability, for sensor placement.

One of the features of the analysis below is that the system equations are retained in linear matrix-second-order form. The greater generality of modern system theory, which is formulated for linear systems in matrix-first-order form, is not required here; in fact, it is by utilizing the special characteristics of the matrix-second-order system equations representative of flexible spacecraft that the results presented below can be obtained.

Problem Definition

The class of spacecraft to which the results derived herein apply is restricted by the following assumptions:

1) The vehicle $\mathcal{V}$ is partly rigid and partly flexible; the rigid part is denoted by $\mathcal{R}$ and the N flexible appendages by $\mathcal{E}_n$ $(n=1,\ldots,N)$. See Fig. 1.

2) The appendages have a linear elastic stress-strain law and experience small deformations only.

3) $\mathcal{V}$ is not spinning, but rotates through small angles with respect to an inertial reference frame.

4) Structural damping, discrete dampers, and other forms of dissipation are not considered in arriving at the main results of this paper (see also Concluding Comments).

5) Rotors, if present at all, are assumed to make a negligible contribution to the angular momentum of the system.

Under these assumptions, the motion equations for $\mathcal{V}$ can be written as follows:

$$m\ddot{\boldsymbol{w}}_c + \sum_{n=1}^{N} \boldsymbol{P}_{n\delta}^T \ddot{\boldsymbol{\delta}}_n = \boldsymbol{F}(t)$$

$$\boldsymbol{I}\ddot{\boldsymbol{\theta}} + \sum_{n=1}^{N} \boldsymbol{H}_{n\delta}^T \ddot{\boldsymbol{\delta}}_n = \boldsymbol{G}(t)$$

$$\boldsymbol{P}_{n\delta}\ddot{\boldsymbol{w}}_c + \boldsymbol{H}_{n\delta}\ddot{\boldsymbol{\theta}} + \boldsymbol{M}_n\ddot{\boldsymbol{\delta}}_n + \boldsymbol{K}_n\boldsymbol{\delta}_n = \boldsymbol{\Upsilon}_{n\delta}(t) \qquad (n=1,\ldots,N) \qquad (1)$$

Here m is the mass of $\mathcal{V}$, and $\boldsymbol{I}$ is the matrix of inertia moments about the mass center c. Small motions of c are represented by $\boldsymbol{w}_c$. Within appendage $\mathcal{E}_n$, there are additional displacement variables corresponding to the deformations arising from structural flexibility. These variables, which are the elements of $\boldsymbol{\delta}_n$, may be discrete (actual displacements at certain points in $\mathcal{E}_n$), or distributed (assumed shape functions, for example) or a combination of the two (hybrid coordinates, as discussed by Likins[3]). When the appendage is a truss, the

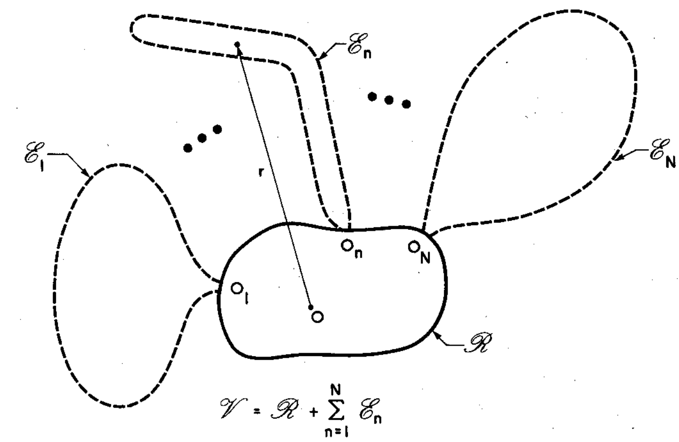

Fig. 1 Flexible space vehicle.

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methods of matrix structural analysis can be applied; when it is more plausibly modeled as a continuum, the method of finite elements[4] is often appropriate. By whatever method, M_n and K_n are the mass and stiffness matrices for $\mathcal{E}_n$ (translation and rotation proscribed at the point where $\mathcal{E}_n$ is attached to $\mathcal{R}$). Then

$$M_n^T = M_n > 0 \qquad K_n^T = K_n > 0 \tag{2}$$

The coefficients $P_{n\delta}$ and $H_{n\delta}$ arise, respectively, from the contributions of the deformations in $\mathcal{E}_n$ to the momentum and angular momentum of $\mathcal{V}$. The total force and torque (about c) on $\mathcal{V}$ are denoted by F and G, and $\Upsilon_{n\delta}(t)$ contains the genralized forces within $\mathcal{E}_n$.

We have not as yet specified how control actions are to be incorporated in Eq. (1), nor what variables are to be considered as the system output. Prior to doing so, some discussion of the characteristic modes for Eq. (1) is required. The first family of modes are the so-called normal modes for each appendage by itself, with its attachment point constrained against both rotation and translation ($w_c \equiv 0 \equiv \theta$). Thus, we set w_c, θ, and $\Upsilon_{n\delta}$ to zero in Eq. (1). The governing equations for these modes are

$$M_n \ddot{\delta}_n + K_n \delta_n = 0 \qquad (n = 1, \ldots, N) \tag{3}$$

In view of Eq. (2), it is well-known[5] that a transformation T_n exists with the following properties (1 stands for the unit matrix):

$$T_n^T M_n T_n = 1 \qquad T_n^T K_n T_n = \Omega_n^2 \tag{4}$$

where $\Omega_n \triangleq \operatorname{diag}\{\Omega_{n1}, \Omega_{n2}, \ldots\}$ contains the natural frequencies for $\mathcal{E}_n$. These modes are called apppendage modes, fixed-base modes, cantilever modes, or constrained modes.[6] Letting $\delta_n = T_n Q_n$, where Q_n now contains the constrained modal coordinates, Eq. (1) reduces to

$$m\ddot{w}_c + \sum_{n=1}^{N} P_n^T \ddot{Q}_n = F(t)$$

$$I\ddot{\theta} + \sum_{n=1}^{N} H_n^T \ddot{Q}_n = G(t)$$

$$P_n \ddot{w}_c + H_n \ddot{\theta} + \ddot{Q}_n + \Omega_n^2 Q_n = \Upsilon_n(t)$$

$$(n = 1, \ldots, N) \tag{5}$$

where

$$P_n \triangleq T_n^T P_{n\delta} \qquad H_n \triangleq T_n^T H_{n\delta} \qquad \Upsilon_n \triangleq T_n^T \Upsilon_{n\delta} \tag{6}$$

We shall further assume that the vehicle possesses symmetry properties so that the translational and rotational motion (w_c and θ) are uncoupled. Thus, $P_n = 0$ for those modes with $H_n \neq 0$. If this symmetry property is not present, the development proceeds in exactly the same manner as that followed below, although the system matrices are somewhat longer. We make the symmetry assumption, which is often reasonable in practice, and retain only those modes that contribute to $\Sigma H_n^T \dot{Q}_n$, directing attention to attitude control. (If translational motion is of interest, the modes contributing to $\Sigma P_n^T \dot{Q}_n$ should be retained instead.) Thus, Eq. (5) becomes

$$I\ddot{\theta} + H^T \ddot{Q} = G(t) \qquad H\ddot{\theta} + \ddot{Q} + \Omega^2 Q = \Upsilon(t) \tag{7}$$

where

$$H \triangleq \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix} \qquad Q \triangleq \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} \qquad \Upsilon(t) \triangleq \begin{bmatrix} \Upsilon_1(t) \\ \vdots \\ \Upsilon_N(t) \end{bmatrix} \tag{8}$$

$$\Omega \triangleq \operatorname{diag}\{\Omega_1, \ldots, \Omega_N\} \tag{9}$$

The formulation must be completed by specifying control variables and output variables, which is done in detail later. For the present, it is more important to note the structure of the system equations regardless of what type of control devices are selected. This form is as follows:

$$M\ddot{q} + Kq = Bu \qquad y = Pq + P'\dot{q} \tag{10}$$

where, from Eq. (7),

$$M \triangleq \begin{bmatrix} I & H^T \\ H & 1 \end{bmatrix} \qquad K \triangleq \begin{bmatrix} 0 & 0 \\ 0 & \Omega^2 \end{bmatrix} \qquad q \triangleq \begin{bmatrix} \theta \\ Q \end{bmatrix} \tag{11}$$

and

$$B \triangleq \begin{bmatrix} B_{rr} & B_{re} \\ 0 & B_{ee} \end{bmatrix} \qquad P \triangleq \begin{bmatrix} P_{rr} & 0 \\ P_{er} & P_{ee} \end{bmatrix}$$

$$P' \triangleq \begin{bmatrix} P'_{rr} & 0 \\ P'_{er} & P'_{ee} \end{bmatrix} \qquad u = \begin{bmatrix} u_r \\ u_e \end{bmatrix} \tag{12}$$

The form of Eq. (12) is justified as follows. Control actions on $\mathcal{R}$ are denoted $u_r(t)$, and those on $\Sigma\mathcal{E}_n$ are denoted $u_e(t)$. Torques on $\mathcal{V}$ may arise due to either, $G(t) = B_{rr}u_r(t) + B_{re}u_e(t)$, but modal inputs can come only from u_e, $\Upsilon(t) = B_{ee}u_e(t)$. The elements of B_{rr}, B_{re}, and B_{ee} are worked out in detail for force actuators in a later section. Measurements present the dual situation. If taken on $\mathcal{R}$, only θ or $\dot{\theta}$ can be sensed, $P_{rr}\theta + P'_{rr}\dot{\theta}$. Measurements on $\Sigma\mathcal{E}_n$, on the other hand, could be of θ, $\dot{\theta}$, Q or $\dot{Q}$, as represented by $P_{er}\theta + P'_{er}\dot{\theta} + P_{ee}Q + P'_{ee}\dot{Q}$.

In later sections, the controllability and observability of Eq. (10) will be examined, based on the authors' theorems[7] for linear matrix-second-order systems.

Results Due to Ohkami and Likins

An earlier paper by Ohkami and Likins[8] dealt with essentially the same issues as raised here. Although they initially considered sensors and actuators both on the appendages and on the main body $\mathcal{R}$, their principal results dealt with controllability and observability when all control devices are on $\mathcal{R}$. Their model was different in one respect, namely, they assumed N nodal bodies connected to $\mathcal{R}$, whereas in this paper N elastic bodies are appended to $\mathcal{R}$. This affects only the meaning of the symbols, however, and their results are immediately applicable to our model, as summarized in the following two theorems.

Theorem 1: Consider the system Eq. (10), and assume that all actuators are on $\mathcal{R}$ ($B_{re} = 0$; $B_{ee} = 0$), and that the Ω_{jn} are distinct. Then the constrained modes are all controllable if and only if

$$H_{jn}^T H_{jn} > 0 \qquad (j = 1, 2, \ldots;\; n = 1, \ldots, N) \tag{13}$$

Note that $H_{jn}\dot{Q}_{jn}$ is the contribution of the jth mode in the nth appendage to the angular momentum of the system. H_{jn} is given by

$$H_{jn} \triangleq \int_{\mathcal{E}_n} \tilde{r}\Phi_{jn} \mathrm{d}m \tag{14}$$

where Φ_{jn} is the jth mode in appendage n. When two or more of the Ω_{jn} are equal (even if they are on different appendages), condition (13) must be generalized. Suppose, for example, that Ω_{21}, Ω_{31}, Ω_{72} are equal, and all other frequencies distinct;

the condition that replaces Eq. (13) for the three modes in question is

$$rk[H_{21}^T \quad H_{31}^T \quad H_{72}^T]=3 \tag{15}$$

The general result for repeated Ω_{jn} should be obvious from this example, and hence the rather elaborate notation required to express it will not be contrived. This generalization was also given by Likins and Ohkami. They proved its necessity and suspected its sufficiency. Indeed, an application of the Jordan-form controllability theorem (e.g., Ref. 9, p. 191) to their controllability matrix demonstrates sufficiency also. An analogous theorem was proved[8] for observability:

Theorem 2: Consider the system Eqs. (10) and (12), and further assume that there are no rate sensors and that all sensors are on $\mathcal{R}$ ($P'=0$; $\dot{P}_{er}=0$; $P_{ee}=0$), and that the Ω_{jn} are distinct. Then the constrained modes are all observable if, and only if,

$$H_{jn}^T H_{jn}>0 \qquad (j=1,2,...;\ n=1,...,N) \tag{16}$$

The duality with Theorem 1 is evident. When the Ω_{jn} are not distinct, Eq. (16) is replaced by the same more general rank condition as for Theorem 1.

These two theorems illustrate the aim of the present paper, which is to derive conditions that are simple, computationally straightforward, and have relatively simple physical interpretation. The criteria of Eqs. (13) and (16) satisfy these requirements when the control devices are on $\mathcal{R}$. To obtain conditions of a similar nature when the sensors and actuators are also on the appendages, we turn to a discussion of unconstrained modes.

Unconstrained Modes (Vehicle Modes)

A theory for the controllability and observability of linear matrix-second-order systems has been developed by the authors.[7] It is clear from the theorems in Ref. 7 that the necessary and sufficient conditions are most succinctly and simply expressed in terms of the natural modes of the system. To apply this rule to flexible spacecraft, the conditions we seek are best expressed in terms of the modes of vibration for the entire vehicle $\mathcal{V}$. For vehicle modes, $\mathcal{R}$ is not constrained in either rotation or translation; hence they will be called unconstrained modes. In the αth unconstrained mode shape, denote the (small) rotation of $\mathcal{R}$ by θ_α; furthermore, let $\phi_\alpha(r)$ be the elastic deformation in the appendages. The total displacement in the αth mode shape then is

$$w_\alpha(r)=-\tilde{r}\theta_\alpha+\begin{cases}\phi_\alpha(r) & (r\epsilon\Sigma\mathcal{E}_n)\\ 0 & (r\epsilon\mathcal{R})\end{cases} \tag{17}$$

Moreover, we can express $\phi_\alpha(r)$ as a superposition of constrained modes:

$$\phi_\alpha(r)=\Sigma_j t_{\alpha jn}\Phi_{jn}(r) \qquad (r\epsilon\mathcal{E}_n) \tag{18}$$

where the $t_{\alpha jn}$ are constant coefficients and Φ_{jn} is the jth mode of appendage n. The sum in Eq. (18) is taken over, however, many constrained modes are retained in appendage n after truncation. The fewer modes retained, the more ϕ_α is in error. It is also convenient to define

$$t_{\alpha n}\triangleq[t_{\alpha 1n} \quad t_{\alpha 2n}...]^T \qquad t_\alpha\triangleq[t_{\alpha 1}^T...t_{\alpha N}^T]^T \tag{19}$$

Equation (18) can then be rewritten

$$\phi_\alpha(r)=\Phi_n(r)t_{\alpha n} \qquad (r\epsilon\mathcal{E}_n) \tag{20}$$

where

$$\Phi_n(r)\triangleq[\Phi_{1n}(r) \quad \Phi_{2n}(r)...] \tag{21}$$

The orthogonality and normality conditions for the Φ_{jn} and the ϕ_α are assumed to be[10]:

$$\int_{\mathcal{E}_n}\Phi_n^T\Phi_n dm=1 \qquad \sum_{n=1}^N\int_{\mathcal{E}_n}\phi_\alpha^T\phi_\beta dm-\theta_\alpha^T I\theta_\beta=\delta_{\alpha\beta}$$

$$\int_{\mathcal{E}_n}\Phi_n^T\mathcal{S}[\Phi_n]dr=\Omega_n^2 \qquad \sum_{n=1}^N\int_{\mathcal{E}_n}\phi_\alpha^T\mathcal{S}[\phi_\beta]dr=\omega_\alpha^2\delta_{\alpha\beta} \tag{22}$$

where $\mathcal{S}$ is a stiffness operator (see, for example, Meirovitch[11]) and ω_α is the natural frequency of the αth unconstrained mode. Substituting Eq. (20) into Eq. (22), and invoking Eq. (21), we learn that

$$t_\alpha^T t_\beta-\theta_\alpha^T I\theta_\beta=\delta_{\alpha\beta} \tag{23}$$

$$t_\alpha^T\Omega^2 t_\beta=\omega_\alpha^2\delta_{\alpha\beta} \tag{24}$$

Furthermore, the angular momentum in the appendages due to deformations in the αth unconstrained mode is proportional to

$$h_\alpha\triangleq\sum_{n=1}^N\int_{\mathcal{E}_n}\tilde{r}\phi_\alpha dm \tag{25}$$

To express this angular-momentum coefficient in terms of the constrained angular-momentum coefficients, Eq. (20) is inserted in Eq. (25):

$$h_\alpha=\sum_{n=1}^N\int_{\mathcal{E}_n}\tilde{r}\Phi_n(r)dm\,t_{\alpha n}=\sum_{n=1}^N H_n^T t_{\alpha n}=H^T t_\alpha \tag{26}$$

upon using the definitions of Eqs. (21), (14), and (19). Finally, we note that the αth unconstrained modal input is

$$\begin{aligned}\gamma_\alpha&\triangleq\sum_{n=1}^N\int_{\mathcal{E}_n}\phi_\alpha^T f(r,t)dr\\&=\sum_{n=1}^N t_{\alpha n}^T\int_{\mathcal{E}_n}\Phi_n^T f dr=\sum_{n=1}^N t_{\alpha n}^T\Upsilon_n=t_\alpha^T\Upsilon\end{aligned} \tag{27}$$

where the definitions of Eqs. (20), (42), and (8) have been used.

The foregoing relationships have been derived to facilitate transformation from constrained to unconstrained modes. In terms of constrained modes, the system equations are Eqs. (10-12). It is known from linear algebra that the properties $M^T=M>0$ and $K^T=K$ imply the existence of transformation T such that

$$T^TMT=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad T^TKT=\begin{bmatrix}0 & 0\\ 0 & \omega^2\end{bmatrix} \tag{28}$$

$$\omega\triangleq\text{diag}\{\omega_1,\omega_2,....\} \tag{29}$$

But we can do much more than state that T exists; it can be shown that T has the explicit form

$$T=\begin{bmatrix}I^{-1/2} & \theta_1 & \theta_2 & ...\\ 0 & t_1 & t_2 & ...\end{bmatrix} \tag{30}$$

where θ_α has the simple physical interpretation of being the rotation of $\mathcal{R}$ for the αth unconstrained mode, and t_α represents the constants in terms of which the unconstrained modes can be expressed as a linear combination of constrained modes. The right-hand sides of Eq. (28) are verified by inserting Eq. (30), and M and K from Eq. (11), in the left-hand sides. In this derivation, the total angular momentum of an unconstrained mode is zero (since there is no torque):

$$h_\alpha+I\theta_\alpha=0 \tag{31}$$

This relation is needed to simplify the expressions.

To convert from constrained to unconstrained modal coordinates, we let $q = T\eta$; in partitioned form,

$$\theta = I^{-1/2}\eta_r + \Sigma\theta_\alpha\eta_\alpha \qquad Q = \Sigma t_\alpha \eta_\alpha \tag{32}$$

where η_r corresponds to the three rigid modes, and η_α is the generalized coordinate associated with the αth unconstrained mode. After substitution of Eq. (32) in Eq. (10), and premultiplication by T^T, the resulting differential equations for $\eta(t)$ are found to be

$$\ddot{\eta}_r = I^{-1/2}(B_{rr}u_r + B_{re}u_e)$$

$$\ddot{\eta}_\alpha + \omega_\alpha^2\eta_\alpha = \theta_\alpha^T B_{rr}u_r + (\theta_\alpha^T B_{re} + t_\alpha^T B_{ee})u_e \tag{33}$$

In a similar manner, the output equation (10b) can be written

$$y = \begin{bmatrix} P_{rr}I^{-1/2} \\ P_{er}I^{-1/2} \end{bmatrix} \eta_r + \Sigma \begin{bmatrix} P_{rr}\theta_\alpha \\ P_{er}\theta_\alpha + P_{ee}t_\alpha \end{bmatrix} \eta_\alpha$$

$$+ \begin{bmatrix} P'_{rr}I^{-1/2} \\ P'_{er}I^{-1/2} \end{bmatrix} \dot{\eta}_r + \Sigma \begin{bmatrix} P'_{rr}\theta_\alpha \\ P'_{er}\theta_\alpha + P'_{ee}t_\alpha \end{bmatrix} \dot{\eta}_\alpha \tag{34}$$

Now that the governing equations have been expressed in terms of system modes (unconstrained modes), the theory of Ref. 7 is applicable.

Controllability

According to Theorem 1 of Ref. 7, the necessary and sufficient conditions for controllability are

$$rk[I^{-1/2}(B_{rr} \quad B_{re})] = 3 \tag{35}$$

$$rk[\theta_\alpha^T B_{rr} \quad \theta_\alpha^T B_{re} + t_\alpha^T B_{ee}] = 1 \qquad (\alpha = 1,2,\ldots) \tag{36}$$

It has been assumed in Eq. (36) that the ω_α are distinct (if not, see below). An alternate statement of these conditions is in terms of the controllability norms, defined by

$$\mathcal{C}_r \triangleq \{\det[I^{-1/2}(B_{rr} \quad B_{re})(B_{rr} \quad B_{re})^T I^{-1/2}]\}^{1/6}$$

$$= \left[\frac{\det(B_{rr}B_{rr}^T + B_{re}B_{re}^T)}{\det I}\right]^{1/6} \tag{37}$$

$$\mathcal{C}_\alpha \triangleq \|\theta_\alpha^T B_{rr} \quad \theta_\alpha^T B_{re} + t_\alpha^T B_{ee}\| \qquad (\alpha = 1,2,\ldots) \tag{38}$$

In the latter definition, the Euclidean norm is intended. Conditions (35) and (36) become

$$\mathcal{C}_r > 0 \qquad \mathcal{C}_\alpha > 0 \qquad (\alpha = 1,2,\ldots) \tag{39}$$

The first condition, $\mathcal{C}_r > 0$, is necessary and sufficient for controllability of the rigid ($\omega^2 = 0$) modes, while the αth unconstrained elastic mode is controllable if and only if $\mathcal{C}_\alpha > 0$. This form of the conditions, Eq. (39), is more useful than Eqs. (35) and (36) because $\mathcal{C}_\alpha$ represents a measure of the degree of controllability of each mode.

The necessary and sufficient conditions for controllability are therefore summarized by the following theorem.

Theorem 3: Consider the system defined by Eq. (10), which can be transformed to Eq. (33), and assume that the ω_α are distinct. The system is controllable if and only if

$$\mathcal{C}_r > 0 \qquad \mathcal{C}_\alpha > 0 \quad (\alpha = 1,2,\ldots)$$

Moreover, the degrees of controllability of the rigid and elastic modes are in the ratios $\mathcal{C}_r : \mathcal{C}_1 : \mathcal{C}_2 : \ldots$.

If the ω_α are not distinct, this theorem can be generalized,[7] although the detailed notation will not be developed here to state the most general case of multiple frequencies. Instead, an example will suffice. Suppose $\omega_1 = \omega_2$, but all the other ω_α are distinct. Then the conditions $\mathcal{C}_1 > 0, \mathcal{C}_2 > 0$ are replaced by the combined condition

$$rk \begin{bmatrix} \theta_1^T B_{rr} & \theta_1^T B_{re} + t_1^T B_{ee} \\ \theta_2^T B_{rr} & \theta_2^T B_{re} + t_2^T B_{ee} \end{bmatrix} = 2 \tag{40}$$

which requires, as before, that $\mathcal{C}_1 > 0$, $\mathcal{C}_2 > 0$, but which also requires that the two rows in Eq. (40) not be proportional.

As a special case of this theorem, force actuators are now considered. We denote by $f(r,t)$ the force per unit volume on $\mathcal{V}$, at r, at time t. Expressed in terms of f, the torque G is

$$G(t) \triangleq \int_{\mathcal{V}} \tilde{r}f(r,t)\,\mathrm{d}r \tag{41}$$

and the modal inputs are given by

$$\Upsilon_n(t) \triangleq \int_{\mathcal{E}_n} \Phi_n^T(r)f(r,t)\,\mathrm{d}r \qquad (n = 1,\ldots,N) \tag{42}$$

where $\mathrm{d}r$ is a volume element at r. Although f will, in general, include disturbance forces as well as control forces, it is consistent without objective of assessing controllability to confine attention to the control forces. If the forces are disturbances rather than controls, the results of this paper can be interpreted as disturbability rather than controllability criteria.

We now position i_r force actuators on $\mathcal{R}$, and i_n force actuators on $\mathcal{E}_n$ ($n = 1,\ldots,N$) as follows: $f_i(t)$ acts at the point a_i, $a_i \in \mathcal{R}$ ($i = 1,\ldots,i_r$), and $f_{ni}(t)$ acts at the point a_{ni}, $a_{ni} \in \mathcal{E}_n$, $[(i = 1,\ldots,i_n), n = 1,\ldots,N]$. In symbols,

$$f(r,t) = \sum_{i=1}^{i_r} f_i(t)\delta(r - a_i) + \sum_{n=1}^{N}\sum_{i=1}^{i_n} f_{ni}(t)\delta(r - a_{ni}) \tag{43}$$

where $\delta(\cdot)$ is the Dirac δ-function. Inserting Eq. (43) into Eqs. (41) and (42), we find

$$G(t) = \sum_{i=1}^{i_r} \tilde{a}_i f_i(t) = \sum_{n=1}^{N}\sum_{i=1}^{i_n} \tilde{a}_{ni} f_{ni}(t) \tag{44}$$

$$\Upsilon_n(t) = \sum_{i=1}^{i_n} \Phi_n^T(a_{ni}) f_{ni}(t) \qquad (n = 1,\ldots,N) \tag{45}$$

The next modeling decision concerns a choice of control variables. We could, for example, make the f_i and f_{ni} constant in magnitude and variable in direction. Instead, we fix the directions of these forces and choose their magnitudes to be control variables

$$f_i(t) = n_i u_i(t) \qquad (i = 1,\ldots,i_r) \tag{46}$$

$$f_{ni}(t) = n_{ni} u_{ni}(t) \qquad [(i = 1,\ldots,i_n);(n = 1,\ldots,N)] \tag{47}$$

where n_i and n_{ni} have unit magnitude. Then

$$G(t) = B_{rr}u_r(t) + B_{re}u_e(t) \tag{48}$$

$$\Upsilon(t) = B_{ee}u_e(t) \tag{49}$$

where

$$B_{rr} \triangleq [a_1 n_1 \ldots \tilde{a}_{i_r} n_{i_r}] \tag{50}$$

$$B_{re} \triangleq ([\tilde{a}_{11}n_{11} \ldots \tilde{a}_{1i_1} n_{1i_1}] \ldots [\tilde{a}_{N1}n_{N1} \ldots \tilde{a}_{Ni_N} n_{Ni_N}]) \tag{51}$$

$$B_{ee}=\begin{bmatrix}[\Phi_1^T(a_{11})n_{11}\ldots\Phi_1^T(a_{1i_1})n_{1i_1}]\ldots & & 0\\ \vdots & & \vdots\\ 0 & & \ldots[\Phi_N^T(a_{N1})n_{N1}\ \ldots\ \Phi_N^T(a_{Ni_N})n_{Ni_N}]\end{bmatrix} \tag{52}$$

$$u_r(t)\triangleq([u_1\ldots u_{i_r}] \tag{53}$$

$$u_e(t)\triangleq([u_{11}\ldots u_{1i_1}]\ldots[u_{N1}\ldots u_{Ni_N}]) \tag{54}$$

We also note that, by utilizing Eq. (20),

$$t_\alpha^T B_{ee}=([\phi_\alpha^T(a_{11})n_{11}\ldots\phi_\alpha^T(a_{1i_1})n_{1i_1}]$$
$$\ldots[\phi_\alpha^T(a_{N1})n_{N1}\ldots\phi_\alpha^T(a_{Ni_N})n_{Ni_N}]) \tag{55}$$

Condition (35) can now be interpreted physically. It is equivalent to

$$rk[B_{rr}\quad B_{re}]=3 \tag{56}$$

because I is nonsingular. A glance at Eqs. (50) and (51) reveals the plausible interpretation that it is necessary and sufficient that three of the torque vectors $\tilde{a}_i n_i[(i=1,\ldots,i_r)$, $\tilde{a}_{ni}n_{ni}[(i=1,\ldots,i_n);\ n=1,\ldots,N)]$ be noncoplanar. It is also clear that insofar as the rigid modes are concerned, it is a matter of indifference whether the actuators are on $\mathcal{R}$ or $\Sigma\mathcal{E}_n$. As for $\mathcal{C}_\alpha$, it is given in this case by

$$\mathcal{C}_\alpha^2\triangleq\sum_{i=1}^{i_r}[\theta_\alpha^T\tilde{a}_i n_i]^2+\sum_{n=1}^{N}\sum_{i=1}^{i_n}[w_\alpha^T(a_{ni})n_{ni}]^2 \tag{57}$$

This form for $\mathcal{C}_\alpha$ has exploited the observation that [see also Eq. (17)]

$$\theta_\alpha^T B_{re}+t_\alpha^T B_{ee}=\theta_\alpha^T\tilde{a}_{ni}n_{ni}+\phi_\alpha^T(a_{ni})n_{ni}$$
$$=w_\alpha^T(a_{ni})n_{ni}\quad[(i=1,\ldots,i_n);\ n=1,\ldots,N)] \tag{58}$$

It is evident that the αth mode is controllable if and only if one of the torques implied by the f_i at a_i has a component along the axis of rotation of the αth mode in $\mathcal{R}$ (namely, θ_α), or if one of the forces f_{ni} at a_{ni} on $\Sigma\mathcal{E}_n$ has a component in the direction of the displacement of the αth mode at a_{ni}, namely, $w_\alpha(a_{ni})$. These results are summarized in the following corollary.

Corollary 3.1: Consider the system defined by Eqs. (10-12), which can be transformed to Eq. (33), with B_{rr}, B_{re}, and $t^T{}_\alpha B_{ee}$ given respectively by Eqs. (50), (51), and (55) for force actuators on both the rigid body $\mathcal{R}$ and the appendages $\Sigma\mathcal{E}_n$, and assume that the ω_α are distinct. The system is controllable if and only if 1) three of the vectors $\tilde{a}_i n_i(i=1,\ldots,i_r)$, $\tilde{a}_{ni}n_{ni}$ $[(i=1,\ldots,i_n);\ n=1,\ldots,N]$ are noncoplanar and 2) $\mathcal{C}_\alpha$, given by Eq. (57), satisfies $\mathcal{C}_\alpha>0(\alpha=1,2,\ldots)$. Moreover, the degrees of controllability of the rigid and elastic modes are in the ratios $\mathcal{C}_r$: $\mathcal{C}_1$:$\mathcal{C}_2$:..., where $\mathcal{C}_r$ is given by Eq. (37).

For some applications, it may be preferable to use torque actuators as well as (or instead of) force actuators. The controllability conditions derived above are now generalized to include torque actuators. Briefly,[12] if there are i_r actuators on $\mathcal{R}$, and i_n actuators on $\mathcal{E}_n$ $(n=1,\ldots,N)$,

$$B_{rr}\triangleq[b_1\ldots b_{i_r}] \tag{59}$$

$$B_{re}=([b_{11}\ldots b_{1i_1}]\ldots[b_{N1}\ldots b_{Ni_N}]) \tag{60}$$

where b_i and b_{ni} are unit vectors defining the axes of action of the torque actuators. Similarly,

$$t_\alpha^T B_{ee}=([\psi_\alpha(a_{11})\ldots\psi_\alpha(a_{1i_1})]\ldots[\psi_\alpha(a_{N1})\ldots\psi_\alpha(a_{Ni_N})]) \tag{61}$$

where $\psi_\alpha(a_{ni})$ is the elastic rotation (due to shear) in mode α, about the direction b_{ni}. (The ith actuator on $\mathcal{E}_n$ is located at a_{ni}.) That is,

$$\psi_\alpha(a_{ni})\triangleq b_{ni}^T\theta_\alpha+\tfrac{1}{2}b_{ni}^T\tilde{\nabla}\phi_\alpha(a_{ni}) \tag{62}$$

where $\tilde{\nabla}$ is the "curl" operator. The controllability of the rigid modes is still indicated by Eq. (37), with B_{rr} and B_{re} given by Eqs. (59) and (60), and the controllability of mode α is found from

$$\mathcal{C}_\alpha^2=\sum_{i=1}^{i_r}(b_i^T\theta_\alpha)^2+\sum_{n=1}^{N}\sum_{i=1}^{i_N}\psi_\alpha^2(a_{ni}) \tag{63}$$

The controllability results are summarized by the following corollary to Theorem 3.

Corollary 3.2: Consider the system defined by Eqs. (10-12), which can be transformed to Eq. (33) with B_{rr}, B_{re}, and $t_\alpha^T B_{ee}$ given respectively by Eqs. (59, 60, and 61), for point-torque actuators on both the rigid body $\jmath$

Corollary 3.2: Consider the system defined by Eqs. (10-12), which can be transformed to Eq. (33) with B_{rr}, B_{re}, and $t_\alpha^T B_{ee}$ given respectively by Eqs. (59, 60, and 61), for point-torque actuators on both the rigid body $\mathcal{R}$ and the appendages $\Sigma\mathcal{E}_n$, and assume that the ω_α are distinct. The system is controllable if and only if 1) three of the vectors $b_i(i=1,\ldots,i_r)$, $b_{ni}[(i=1,\ldots,i_n);\ n=1,\ldots,N]$ are noncoplanar and 2) $\mathcal{C}_\alpha$, given by Eq. (63), satisfies $\mathcal{C}_\alpha>0(\alpha=1,2,\ldots)$. Moreover, the degrees of controllability of the rigid and elastic modes are in the ratios $\mathcal{C}_r$:$\mathcal{C}_1$:$\mathcal{C}_2$:..., where $\mathcal{C}_r$ is given by Eq. (37).

Observability

The treatment of observability is similar in several respects (but not identical) to the above discussion for controllability. According to Theorem 2 in Ref. 7, the observability properties of the system defined by Eqs. (10) and (12) can be inferred from the transformed system given by Eqs. (33) and (34). For observability, it is necessary and sufficient that

$$\mathcal{O}_r>0\qquad\mathcal{O}_\alpha>0\qquad(\alpha=1,2,\ldots) \tag{64}$$

where

$$\mathcal{O}_r\triangleq\left[\frac{\det(P_{rr}^T P_{rr}+P_{er}^T P_{er})}{\det I}\right]^{1/6} \tag{65}$$

$$\mathcal{O}_\alpha^2\triangleq\|\theta_\alpha^T P_{rr}^T\quad\theta_\alpha^T P_{er}^T+t_\alpha^T P_{ee}^T\|^2+\omega_\alpha^2\|\theta_\alpha^T P_{rr}'^T\quad\theta_\alpha^T P_{er}'^T+t_\alpha^T P_{ee}'^T\|^2 \tag{66}$$

It has been assumed for Eq. (66) that the ω_α are distinct. By analogy with the controllability norms, we thus have the observability norms $\mathcal{O}_r$ and $\mathcal{O}_\alpha$, where $\mathcal{O}_r$ indicates the observability of the rigid modes and $\mathcal{O}_\alpha$ the observability of the αth vehicle mode

Theorem 4: Consider the system defined by Eqs. (10-12), which can be transformed to Eqs. (33) and (34), and assume that the ω_α are distinct. The system is observable if and only if

$$\mathcal{O}_r>0\qquad\mathcal{O}_\alpha>0\qquad(\alpha=1,2,\ldots)$$

Moreover, the degrees of observability of the rigid and elastic modes are in the ratios $\mathcal{O}_r$:$\mathcal{O}_1$:$\mathcal{O}_2$:... .

As a special case of this theorem, consider measurements of attitude on $\mathcal{R}$ and deformation on $\Sigma\mathcal{E}_n$. In principle, a sensor might measure a combination of displacement and displacement rate; it can safely be assumed, however, that in practice an individual system output variable y_i is either displacement or displacement rate. At this juncture, all outputs are taken to be displacements, although the results obtained can be easily generalized to accommodate rate measurements. Assume that measurements on $\mathcal{R}$ consist of i_r attitude measurements about the i_r directions specified by the unit vectors $\boldsymbol{b}_i$, $i=1,\ldots,i_r$:

$$y_r(t) \triangleq [\boldsymbol{b}_1^T\boldsymbol{\theta}\ldots\boldsymbol{b}_{i_r}^T\boldsymbol{\theta}]^T \qquad \boldsymbol{P}_{rr}^T \triangleq [\boldsymbol{b}_1\ldots\boldsymbol{b}_{i_r}] \tag{67}$$

Similarly, the measurements on the appendage are assumed to be of elastic deformations; specifically, we assume a measurement at $\boldsymbol{s}_{ni}$ in a direction specified by the unit vector $\boldsymbol{n}_{ni}$ [$(i=1,\ldots,i_n)$; $n=1,\ldots,N$] (Some overlap in the choice of symbols for sensor and actuator parameters is allowed; this should not cause confusion because controllability and observability are considered separately.) Then $\boldsymbol{P}_{er}=\boldsymbol{0}$ and $\boldsymbol{P}_{ee}^T$ has exactly the form of $\boldsymbol{B}_{ee}$, given by Eq. (52), except that the mode shapes are evaluated at the sensor locations $\boldsymbol{s}_{ni}$ instead of at the actuator locations $\boldsymbol{a}_{ni}$. Similarly, $\boldsymbol{P}_{ee}\boldsymbol{t}_\alpha$ has the same form as the transpose of Eq. (55), except that the modal deformation is noted at the sensor locations $\boldsymbol{s}_{ni}$, not at the actuator location $\boldsymbol{a}_{ni}$. The modal observability $\mathcal{O}_\alpha$ is then given by

$$\mathcal{O}_\alpha^2 \triangleq \sum_{i=1}^{i_r} (\boldsymbol{\theta}_\alpha^T\boldsymbol{b}_i)^2 + \sum_{n=1}^{N}\sum_{i=1}^{i_n} [\boldsymbol{\phi}_\alpha^T(\boldsymbol{s}_{ni})\boldsymbol{n}_{ni}]^2 \tag{68}$$

the latter expressions being a consequence of the observation

$$\boldsymbol{t}_\alpha^T\boldsymbol{P}_{ee}^T = ([\boldsymbol{\phi}_\alpha^T(\boldsymbol{s}_{11})\boldsymbol{n}_{11}\ldots\boldsymbol{\phi}_\alpha^T(\boldsymbol{s}_{1i_1})\boldsymbol{n}_{1i_1}]$$
$$\ldots[\boldsymbol{\phi}_\alpha^T(\boldsymbol{s}_{N1})\boldsymbol{n}_{N1}\ldots\boldsymbol{\phi}_\alpha^T(\boldsymbol{s}_{Ni_N})\boldsymbol{n}_{Ni_N}]) \tag{69}$$

Physically, $\mathcal{O}_r>0$ asks that three of the axes about which attitude is sensed, namely, $\boldsymbol{b}_i(i=1,\ldots,i_r)$, be noncoplanar. This agrees with intuition. The condition $\mathcal{O}_\alpha>0$ shows that to observe the αth mode, it may be measured either through its effect on the attitude of $\mathcal{R}$ or through appendage deformations. These findings are summarized by the following corollary.

Corollary 4.1: Consider the system defined by Eqs. (10-12), which can be transformed to Eqs. (33) and (34), with $\boldsymbol{P}_{rr}$ and $\boldsymbol{t}_\alpha\boldsymbol{P}_{ee}^T$ given respectively by Eqs. (67) and (69) for attitude measurements on the rigid body $\mathcal{R}$ and deformation measurements on $\Sigma\mathcal{E}_n$, and assume that the ω_α are distinct. The system is observable if and only if 1) three of the vectors $\boldsymbol{b}_i$ $(i=1,\ldots,i_r)$ are noncoplanar and 2) $\mathcal{O}_\alpha$, given by Eq. (67), satisfies $\mathcal{O}_\alpha>0$ $(\alpha=1,2,\ldots)$. Moreover, the degrees of observability of the rigid and elastic modes are in the ratios $\mathcal{O}_r$: $\mathcal{O}_1$: $\mathcal{O}_2$:..., where $\mathcal{O}_r$ is given by Eq. (65) with $\boldsymbol{P}_{er}=\boldsymbol{0}$.

The usual modifications apply to these statements when the ω_α are indistinct. When the only outputs are attitude measurements on the rigid body $\mathcal{R}$, the necessary and sufficient conditions for observability are 1) three of the vectors $\boldsymbol{b}_i$ $(i=1,\ldots,i_r)$ are noncoplanar and 2) $\boldsymbol{\theta}_\alpha^T\boldsymbol{\theta}_\alpha>0$ $(\alpha=1,2,\ldots)$. (Proof: Condition 2 of the theorem requires that $rk\boldsymbol{P}_{rr}\boldsymbol{\theta}_\alpha=1$; since, by condition 1, $rk\boldsymbol{P}_{rr}=3$, we conclude that $rk\boldsymbol{\theta}_\alpha=1$, i.e. that $\boldsymbol{\theta}_\alpha^T\boldsymbol{\theta}_\alpha>0$.) Similary, when the only outputs are deformation measurements on the appendages $\Sigma\mathcal{E}_n$, the system (specifically, the rigid modes) is unobservable.

Under some circumstances, the measurements made on the appendages may be of local attitude instead of local deformation. One still retains $\boldsymbol{y}_r=\boldsymbol{P}_{rr}\boldsymbol{\theta}$, with $\boldsymbol{P}_{rr}$ as in Eq. (67), but the measurements $\boldsymbol{y}_e$ must be formulated anew. Briefly,[12] $\boldsymbol{t}_\alpha^T\boldsymbol{P}_{ee}^T$ is of the same form as $\boldsymbol{t}_\alpha^T\boldsymbol{B}_{ee}$ given in Eq. (61), with $\boldsymbol{a}_{ni}$ replaced by the sensor location $\boldsymbol{s}_{ni}$, and ψ_α being the total (rigid plus elastic) angular displacement:

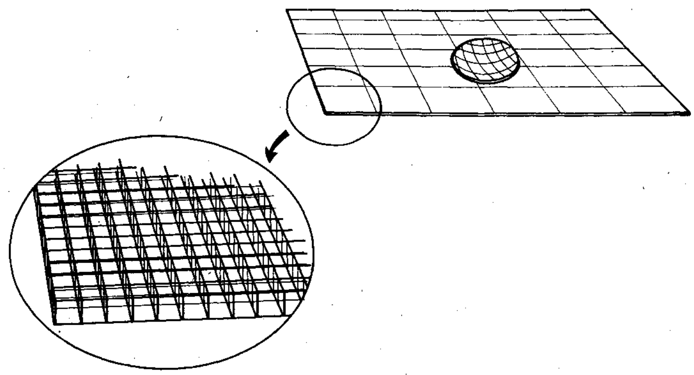

Fig. 2 Purdue model (equivalent distributed parameter model of a dense periodic truss structure).

$$\psi_\alpha(\boldsymbol{a}_{ni}) \triangleq \boldsymbol{b}_{ni}^T\boldsymbol{\theta}_\alpha + \tfrac{1}{2}\boldsymbol{b}_{ni}^T\tilde{\nabla}\boldsymbol{\phi}_\alpha(\boldsymbol{s}_{ni}) \tag{70}$$

Here, $\boldsymbol{b}_{ni}$ is the axis about which the angular displacement is measured. Also, $\boldsymbol{P}_{er}$ is given by the right side of Eq. (60). This leads to the observability

$$\mathcal{O}_\alpha^2 = \sum_{i=1}^{i_r} (\boldsymbol{b}_i^T\boldsymbol{\theta}_\alpha)^2 + \sum_{n=1}^{N}\sum_{i=1}^{i_n} \psi_\alpha^2(\boldsymbol{s}_{ni}) \tag{71}$$

and the following corollary:

Corollary 4.2: Consider the system defined by Eqs. (10-12), which can be transformed to Eqs. (33) and (34) with $\boldsymbol{P}_{rr}$, $\boldsymbol{P}_{er}^T$, and $\boldsymbol{t}_\alpha^T\boldsymbol{P}_{ee}^T$ given by Eqs. (67), (60), and (61), respectively, for attitude measurements on both the rigid body $\mathcal{R}$ and the appendages $\Sigma\mathcal{E}_n$, and assume that the ω_α are distinct. The system is observable if and only if 1) three of the vectors $\boldsymbol{b}_i$ $(i=1,\ldots,i_r)$, $\boldsymbol{b}_{ni}$ $(i=1,\ldots,i_n;\ n=1,\ldots,N)$ are noncoplanar and 2) $\mathcal{O}_\alpha$, given by Eq. (71), satisfies $\mathcal{O}_\alpha>0$ $(\alpha=1,2,\ldots)$. Moreover, the degrees of observability of the rigid and elastic modes are in the ratios $\mathcal{O}_r$: $\mathcal{O}_1$: $\mathcal{O}_2$: ..., where $\mathcal{O}_r$ is given by Eq. (65).

Examples

To illustrate the above results, controllability and observability norms are now calculated for the flexible spacecraft model described in Ref. 13 and shown in Fig. 2. Briefly, it consists of a flexible rectangular structure $\mathcal{E}$, which is approximated by a uniform mass/area and bending stiffness. At the center is located a rigid body $\mathcal{R}$ whose attitude is to be controlled. For this spacecraft, $\boldsymbol{\theta}$ consists of θ_x and θ_y, the attitude angles for $\mathcal{R}$. Elastic deformations are in the z direction only. Suppose, as one illustration, that a force actuator is placed at the point (a_x,a_y) on $\mathcal{E}$, acting in the z direction. It is immediately evident (Corollary 3.1) that the rigid modes are not controllable, $\mathcal{C}_r=0$. At least two force actuators would be required to control θ_x and θ_y independently. For the elastic modes, on the other hand, Eq. (57) reduces, for a unit force, to

$$\mathcal{C}_\alpha = |\phi_\alpha(a_x,a_y)| \tag{72}$$

The function $\mathcal{C}_\alpha$ (a_x,a_y), which is just the rectified mode shape, is plotted for the 15th mode in Fig. 3. As a second illustration of controllability, suppose two point-torque actuators, aligned with the x and y axes, are placed at the point (a_x,a_y). For the actuators whose torque is about the x axis, the vector $\boldsymbol{b}$ is $[1\ 0\ 0]^T$, and for the second actuator $\boldsymbol{b}=[0\ 1\ 0]^T$. So the controllability norm is

$$\mathcal{C}_\alpha = [(\partial\phi_\alpha/\partial x - \theta_{y\alpha})^2 + (\partial\phi_\alpha/\partial y + \theta_{x\alpha})^2]^{1/2} \tag{73}$$

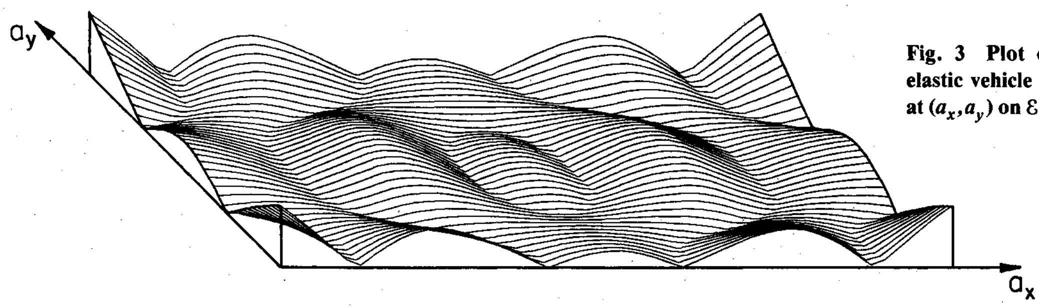

Fig. 3 Plot of controllability norm for 15th elastic vehicle mode; single point-force actuator at (a_x,a_y) on $\mathcal{E}$; $\mathcal{C}_{15} = |\phi_{15}(a_x,a_y)|$.

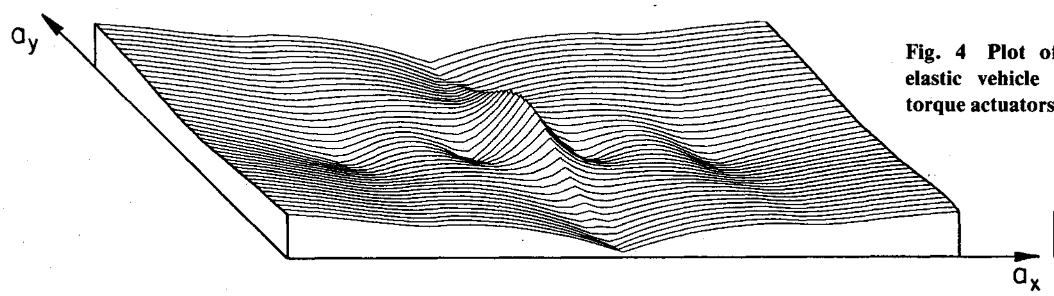

Fig. 4 Plot of controllability norm for 7th elastic vehicle mode; two orthogonal point-torque actuators at (a_x,a_y) on $\mathcal{E}$.

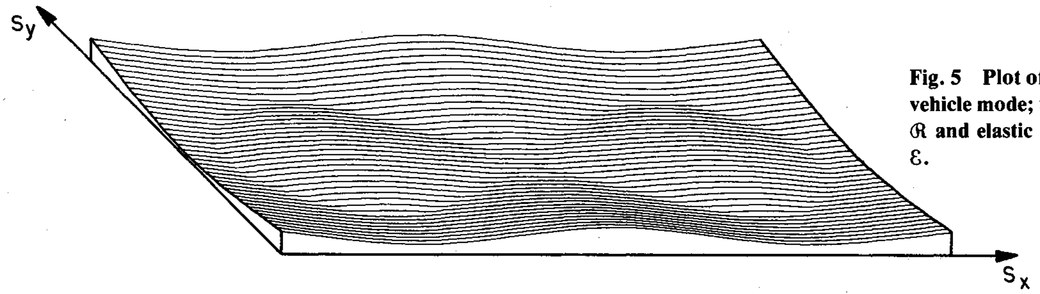

Fig. 5 Plot of observability norm for 8th elastic vehicle mode; two orthogonal attitude sensors on $\mathcal{R}$ and elastic displacement sensor at (s_x, s_y) on $\mathcal{E}$.

This is plotted for $\alpha=7$ in Fig. 4. The controllability norm of the rigid modes is

$$\mathcal{C}_r = (I_x I_y)^{-1/4} \tag{74}$$

after adapting Eq. (37) to the two-dimensional case, and I_x, I_y are the centroidal principal moments of inertia of the vehicle.

To turn to observability, suppose we have two attitude sensors colocated at (s_x,s_y), measuring total (rigid plus flexible) attitude angles about the x and y axes. It can be shown that the observability norms are given by Eq. (74) for the rigid modes and by Eq. (73) for the elastic modes (see Fig. 4). The duality between controllability for torque actuators and observability for attitude sensors is striking. As a final illustration, suppose that two orthogonal attitude sensors are placed on $\mathcal{R}$, and that, in addition, the elastic deflection, given by $\Sigma_\alpha \phi_\alpha(s_x,s_y)\eta_\alpha(t)$, is measured at (s_x,s_y). The latter cannot detect the rigid modes, so the observability norm for these modes is still Eq. (74). The norm for the elastic modes, as required by Corollary 4.1, is given by Eq. (68), which, in this case, reduces to

$$\mathcal{O}_\alpha = [k_1^2(\theta_{x\alpha}^2+\theta_{y\alpha}^2)+k_2^2\phi_\alpha^2(s_x,s_y)]^{1/2} \tag{75}$$

where k_1 and k_2 are the sensor gains for attitude and deformation, respectively. The function $\mathcal{O}_8(s_x,s_y)$ is shown in Fig. 5 when $k_1=k_2=1$.

Concluding Comments

It is clear that, for the class of flexible spacecraft under consideration, one can do much better than quote rank conditions on large unwieldy matrices to resolve questions of controllability and observability. By retaining the matrix-second-order structure, and examining these system properties in the context of unconstrained (or vehicle) modes, the needed conditions have a relatively simple form with a straightforward physical interpretation and minimal need for computational assistance.

From the conditions specified by Theorems 3 and 4, at least n_r sensors and actuators are needed, where n_r is the number of rigid modes to be controlled, assuming no repeated natural frequencies (ω_α). In the theorems, n_r was assumed to be 3; in the examples offered above, $n_r=2$; with freely articulated subbodies, $n_r>3$. Assuming this minimal number of devices is present, the theorems indicate that for controllability or observability to be absent would be most exceptional. One would have to place force actuators exactly at nodes, for example, or have their direction exactly orthogonal to the modal deflection at that point, in order to nullify the required conditions.

Theorems 3-6 allow the determination of controllability and/or observability of individual modes of the system. The ranking of system modes according to the relative degree of modal controllability (observability) has been suggested as a means of model reduction.[14] Modal controllability (observability) has meaning only for a consistent normalization rule for all modes. The consistent normalization applicable to the theorems of this paper is given by Eq. (4). Theorems 3 and 4 suggest such a ranking and also suggest locations for actuators (sensors). Once one has selected the modes one wishes to control (or observe), the functions $\mathcal{C}_\alpha$ (or $\mathcal{O}_\alpha$) for these modes indicate the most beneficial location of the actuators (or sensors). It is not difficult to create meaningful optimization problems based on this theme. It is known that controllability of the open-loop system, as considered here, persists for the closed-loop system also. Nevertheless, final decisions on the positioning of control devices should be postponed until confirmed by closed-loop information.

The presence of small structural damping does not materially change the conclusions in this paper. Slight damping will cause only a slight change in the norms $\mathcal{C}_\alpha$ and

$\mathcal{O}_\alpha$. In principle, the addition of slight damping could change a zero $\mathcal{C}_\alpha$ (or $\mathcal{O}_\alpha$) into a nonzero $\mathcal{C}_\alpha$ (or $\mathcal{O}_\alpha$); this qualitative change is not of practical significance, however, because a small change in modal controllability (observability) norms would not cause significant changes in their ranking.

Acknowledgments

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